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Universal scaling of generalized dimensions on critical strange sets

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Abstract. There exists a global universality for the generalized dimensions $D_q(A)$ on all critical points of phase transitions from period-η-tuplings to chaos in 1D iterative systems. As a consequence of this global universality we propose a new scaling scheme of the generalized dimensions $D_q(A)/D_0(A)$ and the dimension spectra of singularities $f(\alpha(q, A))/f(\alpha(0, A))$, which is also universal, independent of all U-sequences A.

1. Introduction

In the past, studies of quantitative universalities in the chaotic dynamics of 1D iterative systems have achieved many great and important results [1, 2]. These quantitative universalities, such as the scaling factors α_η and the convergent rates δ_η, possess a strong character which depends on the U-sequences of the symbolic dynamics, i.e. quantitatively on the topological property of the map being researched. According to the algorithm of combinatorial theory, the number of admissible words (U-sequences) for the symbolic dynamics of two symbols is infinite [3, 4]. These words are distributed on the entire parameter axis. In [5] we found a global universality for fractal dimensions from the numerical calculation, which does not depend on the U-sequences. The global universal quantity belongs to all the whole systems approximately. We believe that the global universality is important to the thermodynamic formalism of complex systems which has received much attention [6, 7].

In this paper, first of all we shall show the global universality with an improved characteristic equation of arbitrary n-order for D_q instead of that in [5]. As a consequence of this universality we shall propose a new scaling scheme for the generalized dimensions $D_q(A)$ and the dimension spectra $f(\alpha(q, A))$, which is also universal, independent of the U-sequences A. This will reveal a uniform behaviour of infinitely many critical points of phase transitions from periodic attractors to chaos in the Feigenbaum scenario.

2. Form of the global universality

In 1D maps, the important quantitative universalities of period-η-tupling sequences A^{*n} can be implied by the generalized Feigenbaum renormalization group equation of arbitrary η-order [8–12]:

$$\varphi(x)=\alpha_\eta \varphi^\eta(x/\alpha_\eta), \qquad \varphi(0)=1, \qquad \varphi'(0)=0 \tag{1}$$

where η is the basic period associated with the U-sequence A which has the length $|A|=\eta$. The universal scaling factor

$$\alpha_\eta(A)=1/\varphi^\eta(0) \tag{2}$$

has its value depending on A and its sign on the R-parity of A [1, 13].

Now we consider the partition functions [14] or the thermodynamic formalism of dimensions [6, 7, 15] for the critical strange sets which hereafter refer to the period-η-tupling attractors (Feigenbaum-type attractors) formed on the critical points of transitions to chaos. The period-η-tupling attractor has the points of orbit on the nth multifurcation level. On the basis of a rule of symbolic dynamics [16] we can make the arbitrary periodic points be ordered on the coordinate space. It is known from the ordering rule that an attractor consists of η^{n-1} point-clusters on the nth level. We take the two adjacent end points as a minimal covering of the cluster, then the length of the σth covering would be [16]

$$d_n(\sigma)=x_\sigma - x_{\sigma+\eta^{n-1}} \tag{3}$$

where x_σ is the σth iterate of the initial point $x=0$ with respect to the generalized Feigenbaum renormalization group function φ satisfying equation (1), i.e. $x_\sigma=\varphi^\sigma(0)$. *If we give each cluster an equal probability measure $1/\eta^{n-1}$, then the partition functions* of generalized dimensions of the critical strange sets are [17, 18]

$$\Gamma_n(q,\tau,\eta)=\sum_{\sigma=1}^{\eta^{n-1}} (1/\eta^{n-1})^q |d_n(\sigma)|^{-\tau} \tag{4}$$

where q and τ are real and the limit cases lead to

$$\lim_{n\to\infty} \Gamma_n(q,\tau,\eta)=1 \tag{5}$$

only if the expression

$$D_q=\tau(q)/(q-1) \tag{6}$$

holds. Of course the generalized dimensions D_q of the critical strange sets depend only on the covering $d_n(\sigma)$ as the above probability measures have been given. Thus they will totally describe the geometrical complexity of the sets when η changes. However, D_q still depends on the U-sequences. We now decompose the summation into η parts:

$$\sum_{\sigma=1}^{\eta^{n-1}} d_n(\sigma)=\sum_{\nu=0}^{\eta-1}\sum_{\sigma=1}^{\eta^{n-2}} d_n(\eta\sigma-\nu) \tag{7}$$

and use the renormalization group equation (1), then the relation between the coverings of nth and $(n-1)$th levels of the attractor is obtained as [5]

$$\sum_{\sigma=1}^{\eta^{n-1}} d_n(\sigma)=\alpha_\eta^{-1}\sum_{\nu=0}^{\eta-1}\sum_{\sigma=1}^{\eta^{n-2}} A_\eta^\nu(\sigma) d_{n-1}(\sigma) \tag{8}$$

$$A_\eta^\nu(\sigma)=\prod_{\lambda=1}^{\nu} \Phi^{-1}(x_{\eta\sigma+\lambda-\nu}) \tag{9}$$

where $\Phi^{-1}=(\varphi^{-1})'$, $\nu=0,1,2,\ldots,\eta-1$ and $A_\eta^0(\sigma)=1$, $\forall\sigma$.

Repeatedly use (7) and (8) $n-1$ times until $\sigma=1$ and let $d_1(1)=x_1-x_2$ be a renormal scale, the partition function with $n-1$ replaced by n would be

$$\left(\frac{1}{\eta}\right)^{nq}|\alpha_\eta|^{n\tau}\sum_{\nu_1=0}^{\eta-1}\cdots\sum_{\nu_n=0}^{\eta-1}\left|\prod_{i=1}^{n}A_\eta^{\nu_i}\left(\eta^{n-i}-\sum_{j=i+1}^{n}\nu_j\eta^{j-i-1}\right)\right|^{-\tau}=1 \tag{10}$$

where n is sufficiently large. This result coincides with the thermodynamic formalism of dimensions [6, 7]. In order to make (10) clear, we introduce

$$\xi=|\alpha_\eta|^\tau \tag{11}$$

$$\rho_\eta^{\nu_1\cdots\nu_n}=1-\log\left|\prod_{i=1}^{n}A_\eta^{\nu_i}\left(\eta^{n-i}-\sum_{j=i+1}^{n}\nu_j\eta^{j-i-1}\right)\right|\Big/(n\log|\alpha_\eta|). \tag{12}$$

Then from (10) we obtain a characteristic equation of arbitrary n-order for the generalized dimensions

$$\left(\frac{1}{\eta}\right)^{nq}\sum_{\nu_1=0}^{\eta-1}\cdots\sum_{\nu_n=0}^{\eta-1}\xi^{n\rho_\eta^{\nu_1\cdots\nu_n}}-1=0. \tag{13}$$

It should be emphasized that the coefficients $\rho_\eta^{\nu_1\cdots\nu_n}$ do not depend on q. When considering the nth level of the attractor, we can obtain the generalized dimension $D_q(n)$ from (6) and (11)-(13):

$$D_q(n)=\frac{\log\xi_\eta(q,n)}{(q-1)\log|\alpha_\eta|} \tag{14}$$

where $\xi_\eta(q,n)$ is the root of the transcendental equation (13).

We can see in the appendix and [5]

$$\xi_\eta(q,n)\sim\eta^{(q-1)\beta_q}+\mathrm{O}(\varepsilon) \tag{15}$$

where $\mathrm{O}(\varepsilon)$ is sufficiently small, β_q is universal for all U-sequences A (or for all critical strange sets). This shows that

$$D_q(A)\log_\eta|\alpha_\eta(A)|=\beta_q \tag{16}$$

is a global universality independent of the U-sequences A. The curve β_q–q can be given in figure 1 through the numerical calculation. Obviously β_q is monotonically

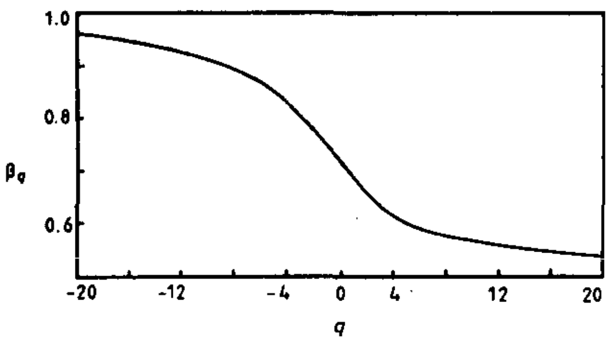

Figure 1. The curve β_q–q for the map $f(x)=1-a|x|^z$ $(z=2)$.

decreasing when q increases. The limit cases should be

$$\beta_{-\infty}=1 \tag{17}$$

$$\beta_{+\infty}=\tfrac{1}{2} \tag{18}$$

because $D_{-\infty}=\log\eta/\log|\alpha_\eta|$ and $D_{+\infty}=\log\eta/\log|\alpha_\eta|^2=\frac{1}{2}D_{-\infty}$ [14, 19].

3. Universal scaling of critical strange sets

From (16) we have

$$D_q(A)/D_0(A)=\beta_q/\beta_0 \tag{19}$$

where β_q/β_0 is a constant for a given q. This relation motivates us to seek a new scaling so that the infinitely many curves of the generalized dimensions D_q–q for the critical strange sets [14, 19] will be able to be scaled as a new universal scaling curve. It is possible to do this if we choose a new variable

$$s_q=D_q/D_0. \tag{20}$$

Thus s_q would be independent of the U-sequences A. The universal scaling curve s_q–q can be immediately obtained from the numerical calculation. It is given in figure 2. It scaled all curves D_q–q (see e.g. figure 3 [19]) into a new one. The universal scaling curve s_q–q has two limit points

$$s_{-\infty}=\beta_{-\infty}/\beta_0=1/\beta_0 \tag{21}$$

$$s_{+\infty}=\beta_{+\infty}/\beta_0=1/(2\beta_0) \tag{22}$$

and a central symmetric point

$$s_0=1. \tag{23}$$

It is emphasized that the scale D_0 in (20) is not unique; from (16) we can choose any other D_{q^*} with a fixed q^* instead of it.

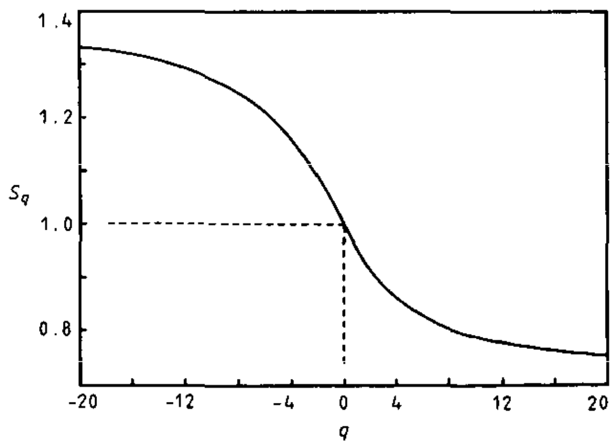

Figure 2. The universal scaling curve of generalized dimensions s_q–q for all critical strange sets ($z=2$).

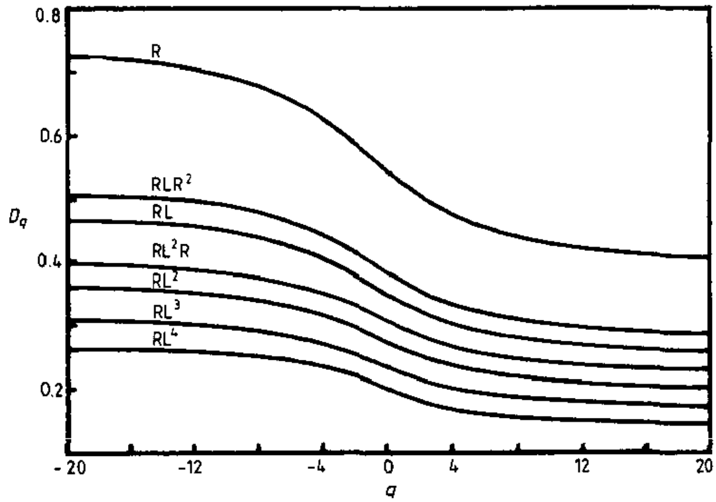

Figure 3. The curves D_q-q for different critical strange sets taken from [19] ($z=2$).

Obviously, paralleling (20) we should consider the scaling of the dimension spectra $f(\alpha)$ [14]. Both $f(\alpha)$ and D_q have no essential difference in describing the global character except for the Legendre transform with which they are linked:

$$\alpha = \frac{\mathrm{d}}{\mathrm{d}q}((q-1)D_q) \qquad f(\alpha) = q\alpha - (q-1)D_q. \tag{24}$$

So the infinitely many curves $f(\alpha(q, A))$-$\alpha(q, A)$ which depend on the U-sequences A would be similarly scaled to a new universal scaling curve. To do this we should note that $f(\alpha(0, A))$ are the maxima of the curves $f(\alpha)$-α while

$$f(\alpha(0, A)) = \alpha(0, A) = D_0(A) \tag{25}$$

and $f(\alpha)=0$ corresponds to α values

$$\alpha_{\max} = \alpha(-\infty, A) = D_{-\infty}(A) \tag{26}$$

$$\alpha_{\min} = \alpha(+\infty, A) = D_{+\infty}(A) = \tfrac{1}{2}D_{-\infty}(A) \tag{27}$$

then we can choose the particular reference point (25) as the scale of the dimension spectra. The new universal scaling curve $f(\alpha(q, A))/f(\alpha(0, A))$-$\alpha(q, A)/\alpha(0, A)$ which is independent of the U-sequences A is plotted in figure 4. It also merges various curves $f(\alpha)$-α for different U-sequences A in figure 5 into a new one.

Finally, we consider the 'free energy' of the critical strange sets. According to [15], the thermodynamic functions s_q or D_q, and τ belong to the microcanonical ensemble for the system of dimension. We can transfer it to the canonical ensemble by the free energy F which is defined as

$$\eta^{-(n-1)F(\theta)} = \sum_{\sigma=1}^{\eta^{n-1}} |d_n(\sigma)|^{-\tau} \tag{28}$$

if n approaches infinity, F becomes independent of n. The relationship between the microcanonical ensemble and canonical ensemble is [15]

$$q = -F(\theta) \qquad \tau = -\theta. \tag{29}$$

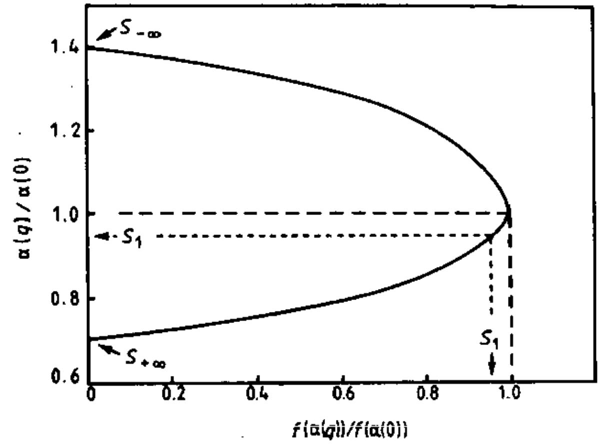

Figure 4. The universal scaling curve of spectra $f(\alpha(q, A))/f(\alpha(0, A))$–$\alpha(q, A)/\alpha(0, A)$ for all critical strange sets ($z=2$). Notice that $f(\alpha(1, A))/f(\alpha(0, A))=\alpha(1, A)/\alpha(0, A)=D_1(A)/D_0(A)=s_1$.

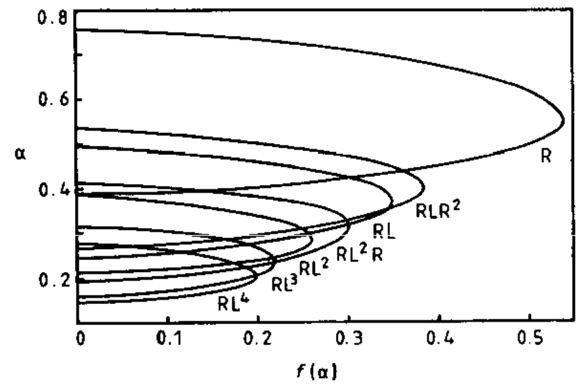

Figure 5. The curves $f(\alpha)$–α for different critical strange sets taken from [19] ($z=2$).

From (16) and (29) the free energy $F(\theta)$ of the critical strange sets satisfies

$$\theta=\beta_{-F(\theta)}(F(\theta)+1)/\log_\eta|\alpha_\eta|. \tag{30}$$

The curve of free energy $F(\theta)$–$\theta \log_\eta|\alpha_\eta|$ in figure 6 shows that it is monotonically increasing and diverging with θ along the line of slope 1 at $+\infty$ and along the line of slope 2 at $-\infty$. It is a global curve when the transversal axis is scaled by different constants.

From the above we can see that these scaling curves are satisfactory; it has shown that the phase transitions from periodic points to the chaotic region in the infinitely many critical strange sets have a global and uniform property. The scaling properties will be brought into any thermodynamic formalism about 1D iterative systems in which the generalized Feigenbaum renormalization group equation plays an important role. We beleive that the new variable s_q will be an important thermodynamic function belonging to all the systems. Moreover, the dependences of those characteristic curves on the degeneracy z of the critical point in the map $f(x)=1-a|x|^z$ $(z>1)$ are very

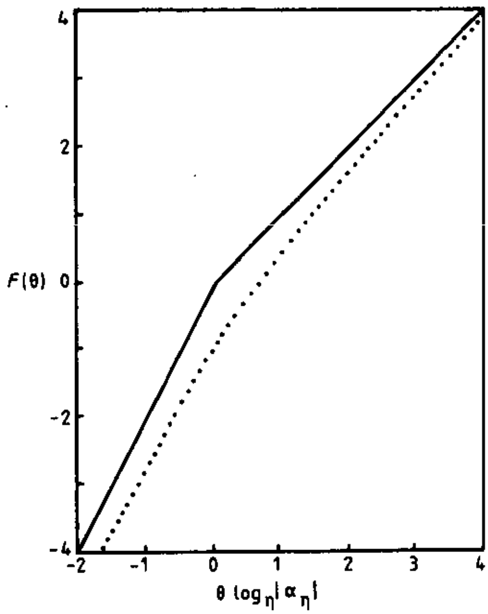

Figure 6. The curve $F(\theta)$-$\theta \log_\eta|\alpha_\eta|$(dotted line, $z=2$).

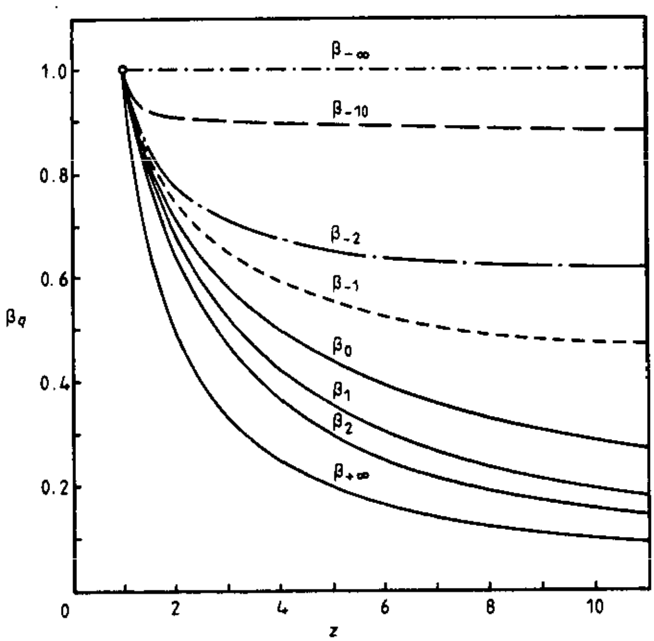

Figure 7. The curves $\beta_q(z)$-z for $q=-\infty, -10, -2, -1, 0, 1, 2, +\infty$.

interesting:

(i) $$D_q(z, A)\log_\eta|\alpha_\eta(z, A)| = \beta_q(z) \tag{31}$$

where the universal functions $\beta_q(z)$ have extrema

$$\beta_{-\infty}(z) = 1 \tag{32}$$

$$\beta_{+\infty}(z) = 1/z \tag{33}$$

because $D_{-\infty} = \log\eta/\log|\alpha_\eta|$ and $D_{+\infty} = \log\eta/\log|\alpha_\eta|^z = (1/z)D_{-\infty}$ [20–22]. The characteristics of $\beta_q(z)$ are shown in figures 7 and 8.

(ii) $$s_q(z) = \frac{D_q(z, A)}{D_0(z, A)} = \frac{\beta_q(z)}{\beta_0(z)} \tag{34}$$

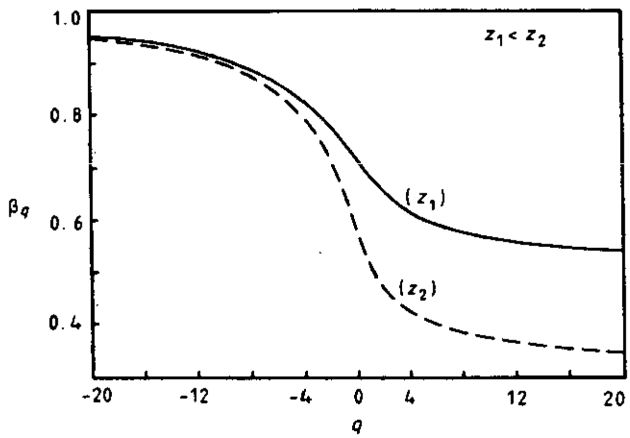

Figure 8. The characteristic of the curves $\beta_q(z)$–q for different values of z (we only show $z_1=2$ and $z_2=3$ as an example).

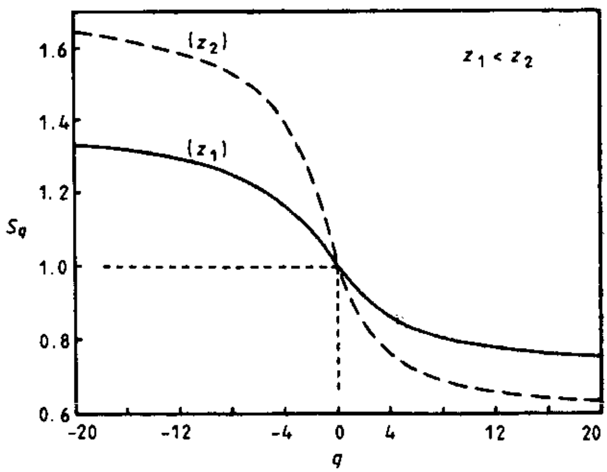

Figure 9. The characteristic of the universal scaling curves $s_q(z)$–q for different values of z ($z_1=2$ and $z_2=3$ as an example).

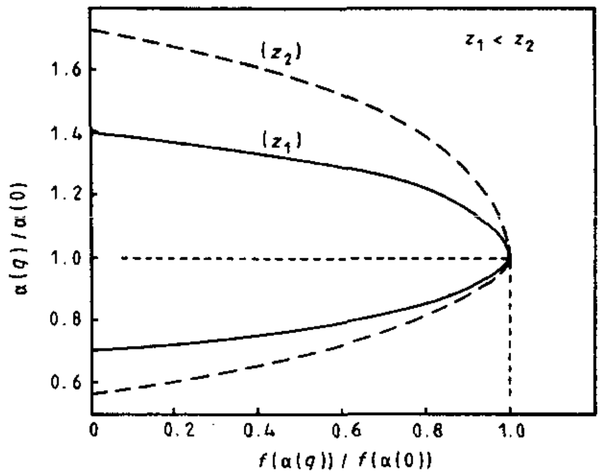

Figure 10. The characteristic of the universal scaling curves $f(\alpha(q, z, A))/f(\alpha(0, z, A))$-$\alpha(q, z, A)/\alpha(0, z, A)$ for different values of z ($z_1 = 2$ and $z_2 = 3$ as an example).

the limit points of the universal scaling curves $s_q(z)$-q (see figure 9) are

$$s_{-\infty}(z) = \frac{1}{\beta_0(z)} \tag{35}$$

$$s_{+\infty}(z) = \frac{1}{z\beta_0(z)}. \tag{36}$$

(iii) The characteristic of the universal scaling curves $f(\alpha(q, z, A))/f(\alpha(0, z, A))$-$\alpha(q, z, A)/\alpha(0, z, A)$ for different values of z is shown in figure 10.

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Appendix Range of $\rho_\eta^{\nu_1 \cdots \nu_n}$ and numerical verification for the global universality

First, we focus our attention on the coefficients $\rho_\eta^{\nu_1 \cdots \nu_n}$. Using (9) and the property of derivative of inverse function

$$\Phi^{-1}(x_\sigma) = 1/\Phi(x_{\sigma-1}) \qquad \Phi = \varphi' \tag{A1}$$

we obtain from (12)

$$\rho_\eta^{\nu_1 \cdots \nu_n} = 1 + \log \left| \prod_{i=1}^{n} \prod_{\lambda_i=1}^{\nu_i} \Phi\left(x_{\eta^{n-i+1} - \lambda_i - \sum_{j=i+1}^{n} \nu_j \eta^{j-i}}\right) \right| \Big/ (n \log |\alpha_\eta|)$$

$$\nu_i = 0, 1, 2, \ldots, \eta - 1 \qquad \forall i. \tag{A2}$$

It readily leads to

$$\min\{\rho_\eta^{\nu_1\cdots\nu_n}\}=\rho_\eta^{0\cdots 0}=1. \tag{A3}$$

Noting the fact that

$$\prod_{\lambda=1}^{\eta-1}\Phi(x_\lambda)=\alpha_\eta \tag{A4}$$

then another extremum of the coefficients $\rho_\eta^{\nu_1\cdots\nu_n}$ yields

$$\max\{\rho_\eta^{\nu_1\cdots\nu_n}\}=\rho_\eta^{\eta-1\cdots\eta-1}=2. \tag{A5}$$

Thus we have

$$1\leqslant\rho_\eta^{\nu_1\cdots\nu_n}\leqslant 2. \tag{A6}$$

In order to obtain the global universality, one should solve the characteristic equation (13) to find the 'exact' value of D_q with the limit $n\to\infty$, but this is impossible in operation. However, as indicated in [11], we only need to take D_q as the expression $nDq(n)-(n-1)D_q(n-1)$ which converges to the 'exact' value very rapidly in a finite n. So in fact we do not need to take a sufficiently large n. In addition, for $q=1$ of (14) we should take the limits $q\to 1^-$ and $q\to 1^+$ to approach D_1.

From the numerical calculation we can see:

(i) For the same basic period η, the solutions $\xi_\eta(q, n)$ of equation (13) (of arbitrary integer n) corresponding to different U-sequences A are almost the same; this implies that

$$\beta_q(n)=\log_\eta\xi_\eta(q, n)/(q-1) \tag{A7}$$

is independent of the U-sequences A of same η.

(ii) For the different U-sequences A of arbitrary basic period η:

$$\beta_q\sim 1/\langle\rho\rangle_q \tag{A8}$$

holds with a high precision, where $\langle\rho\rangle_q$ is the generalized arithmetic averages of $\rho_\eta^{\nu_1\cdots\nu_n}$:

$$\begin{aligned}\langle\rho\rangle_q&=\left(\frac{1}{\eta^n}\sum_{\nu_1=0}^{\eta-1}\cdots\sum_{\nu_n=0}^{\eta-1}(\rho_\eta^{\nu_1\cdots\nu_n})^q\right)^{1/q} \qquad (q\neq 0)\\ \langle\rho\rangle_0&=\left(\prod_{\nu_1=0}^{\eta-1}\cdots\prod_{\nu_n=0}^{\eta-1}\rho_\eta^{\nu_1\cdots\nu_n}\right)^{1/\eta^n} \qquad (q=0).\end{aligned} \tag{A9}$$

These show that β_q is universal, i.e. independent of the U-sequences A and basic period η, thus the global universality (16) is obviously valid.

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